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CONTINUAL MECHANICS OF MONODISPERSE SUSPENSIONS. RHEOLOGICAL EQUATIONS OF STATE FOR SUSPENSIONS OF MODERATE CONCENTRATION

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Rheological relationships linking mean and moment stresses and, also, the force and moment of interphase reaction in a macroscopic flow of small solid sphere suspension with the kinematic characteristics of the flow are derived. This makes it possible to close the system of equations of suspension hydrodynamics. Coefficients of viscosity and of moment viscosity of a suspension are obtained and calculated.

The equations of conservation of mass, momentum and moment of momentum of suspension and of its phases, considered (from the macroscopic point of view) to be coexistent continuous media, were formulated in a general form in [1]. These equations contain unknown vectors and tensors which define the interaction between the considered continuous media and, also, stresses and moment stresses appearing when these are in motion. To close the equations of conservation it is necessary to express all these quantities in terms of unknown variables of these equations (mean concentration of suspension, pressure in the fluid phases, and phase velocities). This problem is the second of the fundamental problems of hydromechanics of suspensions indicated in [1].

Here this problem is solved with the use of a kind of self-consistent field theory, which is essentially an extension and generalization of methods developed in [2 – 7]. Expressions for all of the quantities mentioned above are derived. They can be considered to be rheological equations of state for suspensions. Expressions for the various coefficients of these equations and their dependence on parameters of phases and on the flow frequency spectrum are also considered.

1. It was shown in [1] that in order to express the unknown terms of equations of conservation in terms of observable variables which define the macroscopic motion of suspension it is necessary to determine similar expressions for the mean stresses acting at the surface of an individual particle. In this case averaging is effected over a considerable number of particles under identical conditions. Hence it is natural to begin the analysis by introducing the statistical ensemble of particles (more precisely: their admissible configurations in space) and specifying the method of averaging over the ensemble.

First, as in [8], we introduce the function of distribution for one of the N spherical particles of the system, such that the probability of finding the center of that particle in a volume element $d\mathbf{r}$ close to point $\mathbf{r}$ is $\varphi(t, \mathbf{r})d\mathbf{r}$. We also introduce the conditional unary distribution function $\varphi(t, \mathbf{r} \mid \mathbf{r}')$ of finding the center of particle at point

$\mathbf{r}$ on condition that the center of another particle lies at point $\mathbf{r}'$. Actually the last quantity represents a binary (two-particle) distribution function.

We similarly introduce the unconditional distribution function $\varphi(t, C_N)$ for the ensemble of N particles, where C_N denotes the set of vectors $\mathbf{r}^{(j)}$ $(j=1, 2, \ldots, N)$ which determine the position of centers of all particles and, also, the related conditional distribution functions $\varphi(t, C_{N-1} \mid \mathbf{r})$ and $\varphi(t, C_{N-2} \mid \mathbf{r}, \mathbf{r}')$ associated with the situation in which it is known that the centers of one or two spheres are, respectively, at $\mathbf{r}$ or $\mathbf{r}$ and $\mathbf{r}'$. We have the relationships

$$\varphi(t, C_N) = \varphi(t, \mathbf{r})\,\varphi(t, C_{N-1} \mid \mathbf{r}) \qquad (1.1)$$
$$\varphi(t, C_{N-1} \mid \mathbf{r}') = \varphi(t, \mathbf{r} \mid \mathbf{r}')\,\varphi(t, C_{N-2} \mid \mathbf{r}, \mathbf{r}')$$

Averaging over the ensemble is carried out using definitions

$$\langle F \rangle = \langle F \rangle_N = \int F(t, \mathbf{r}; C_N)\,\varphi(t, C_N)\,dC_N \qquad (1.2)$$
$$\langle F \rangle' = \langle F \rangle'_{N-1} = \int F(t, \mathbf{r}; C_N)\,\varphi(t, C_{N-1} \mid \mathbf{r}')\,dC_{N-1}$$
$$\langle F \rangle'' = \langle F \rangle''_{N-2} = \int F(t, \mathbf{r}; C_N)\,\varphi(t, C_{N-2} \mid \mathbf{r}', \mathbf{r}'')\,dC_{N-2}$$

where F is an arbitrary function of t and $\mathbf{r}$ which depends on the configuration of particles C_N, and dC_N is a volume element in a $3N$-dimensional space formed by the radius vectors of centers of N particles. It is evident that $\langle F \rangle$ is a function of t and $\mathbf{r}$, $\langle F \rangle'$ is a function of t, $\mathbf{r}$ and $\mathbf{r}'$, while $\langle F \rangle''$ is a function of t, $\mathbf{r}$, $\mathbf{r}'$ and $\mathbf{r}''$. It is clear that

$$\langle F \rangle = \int \langle F \rangle'(t, \mathbf{r}; \mathbf{r}')\,\varphi(t, \mathbf{r}')\,d\mathbf{r}' \qquad (1.3)$$
$$\langle F \rangle' = \int \langle F \rangle''(t, \mathbf{r}; \mathbf{r}', \mathbf{r}'')\,\varphi(t, \mathbf{r}'' \mid \mathbf{r}')\,d\mathbf{r}''$$

In what follows the term mean quantities defines quantities averaged over the ensemble in accordance with (1.2). Averaging over a small physical volume (of the kind used in [1]) will be defined separately. The averaging operation (1.2) and differentiations with respect to the independent variable $\mathbf{r}$ commutate, i.e.

$$\langle \partial F / \partial r_i \rangle = (\partial / \partial r_i)\,\langle F \rangle \qquad (1.4)$$

and similar commutative relationships are valid for conditional averages in (1.2).

For functions F whose characteristic variation time is smaller than the time scale T of distribution functions the following commutation relationship

$$\left\langle \frac{\partial F}{\partial t} \right\rangle = \int \frac{\partial F}{\partial t}(t, \mathbf{r}; C_N)\,\varphi(t, C_N)\,dC_N = \frac{\partial}{\partial t}\int F(t, \mathbf{r}; C_N)\varphi(t, C_N)\,dC_N + \qquad (1.5)$$
$$\int F(t, \mathbf{r}; C_N)\,\frac{\partial \varphi}{\partial t}(t, C_N) \approx \frac{\partial}{\partial t}\langle F \rangle$$

is approximately valid.

Variation of the distribution function with time is determined by the redistribution of a great number of particles of the system. Hence scale T must be considerably greater than the time scale of microscopic quantities which define the flow at the level of individual particles. The latter is determined by the rate of change of the hydrodynamic environment in the neighborhood of a single particle, hence formula (1.5) is always satisfied for all of the indicated quantities. The time scale τ of related averaged quantities is determined by the rate of change of boundary conditions imposed on the suspension. Below, when considering unsteady flows, we assume that

$$T \gg \tau \tag{1.6}$$

When this inequality is violated ($T \sim \tau$), the variation of the flow with time may be considered to be a flow process and, as shown below, the instability of the suspension macroscopic motion virtually does not affect the shaping of its rheological properties. A more detailed analysis of the relation between time scales is given in [1].

We further assume that the space scale L of the macroscopic flow is considerably greater than the mean distance between adjacent suspended particles, i.e. that

$$L \gg l \sim a\rho^{-1/3} \tag{1.7}$$

where $\rho(t, \mathbf{r})$ is the volume concentration and a the radius of suspended particles. The latter is the necessary condition of applicability of the method of mechanics of continuous media for defining macroscopic behavior of suspensions [1], and means that the introduced ensemble is microscopically homogeneous, i.e. it is possible to separate a volume containing a great number of particles within which function $\varphi(t, \mathbf{r})$ is virtually independent of $\mathbf{r}$. However in macroscopic volumes, i.e. at distances commensurate with the L-scale, the distribution function $\varphi(t, \mathbf{r})$ is generally nonuniform. The relation between $\varphi(t, \mathbf{r})$ and the mean volume $\rho(t, \mathbf{r})$ and the denumerable concentration $n(t, \mathbf{r})$ of particles, derived in [1], is determined by

$$n(t, \mathbf{r}) = N\varphi(t, \mathbf{r}), \qquad \rho(t, \mathbf{r}) = {}^4/_3\,\pi a^3 N\varphi(t, \mathbf{r}) \tag{1.8}$$

We introduce the condition of "correlation attenuation" for any random function $F(t, \mathbf{r}; C_N)$ which substantially depends on the position of many particles. Namely, we assume that the conditional averages $\langle F\rangle'$ and $\langle F\rangle''$ calculated for configurations in which the centers of one or two particles fixed at points $\mathbf{r}$ or $\mathbf{r}'$ and $\mathbf{r}''$ asymptotically tend to the unconditional average $\langle F\rangle$ with unbounded increase of distances between point $\mathbf{r}$ and the centers of fixed particles. Thus

$$\lim \langle F\rangle'' = \lim \langle F\rangle' = \langle F\rangle, \qquad |\mathbf{r} - \mathbf{r}'| \sim |\mathbf{r} - \mathbf{r}''| \to \infty \tag{1.9}$$

Physically this means that, for instance, the mean velocity or pressure of fluid at some point is virtually independent of the state of particles lying fairly far away from that point. Actually condition (1.9) becomes satisfied when $|\mathbf{r} - \mathbf{r}'| \sim |\mathbf{r} - \mathbf{r}''|$ exceed the "interaction scale" L_i in a system which, without loss of generality, can be considered a quantity of the order of L. Condition (1.9) is satisfied in the majority of cases of practical interest, except that of motion in narrow channels whose transverse linear dimension is of the order of L_i or smaller. In the latter case the correlations which are dependent on the long-range interaction between suspended particles are considerable, and may, for instance, result in the formation of the "plug" type mode of flow in the channel.

Inequalities (1.7) and (1.9) limit to a certain extent the class of systems to which the analysis described below is applicable. They are, however, valid for a fairly wide particular class of suspensions and their flow.

Let us introduce the basic simplifying assumption about the structure and properties of the considered ensemble, namely, that the distribution of particles of the suspension is random, so that

$$\varphi(t, \mathbf{r} \mid \mathbf{r}') = \begin{cases} 0, & |\mathbf{r} - \mathbf{r}'| < 2a \\ \varphi(t, \mathbf{r}), & |\mathbf{r} - \mathbf{r}'| > 2a \end{cases} \tag{1.10}$$

As shown in [8], the assumption of total randomness of possible configurations C_N of a cloud of particles (on the equal statistical weight of various configurations) can be strictly satisfied only in conditions of uniform flow and initial random distribution of suspended particles. In all other cases, owing to hydrodynamic interaction, a nontrivial binary distribution function, different from (1.10) and dependent on the properties of flow (an example of successive computation of such function for a simple shear flow appears in [9]), must be obtained. However, from the heuristic point of view, the concept of the system with random uncorrelated distribution of particles may be considered as the first approximation to actual systems with a specific microstructure defined by the pattern of hydrodynamic interaction between particles in this or that flow. It can be assumed that the results of calculations based on the hypothesis about the absence of correlation between the position of adjacent particles (except the condition of inpenetrability of particles implied by (1.10) may prove approximately valid also for a wide class of motions, in spite of the evident inaccuracy of the hypothesis itself. The widely used assumption, confirmed by experiments, that rheological parameters of suspension (e.g., its effective viscosity) are reasonably general, in the sense that they are virtually the same for flows and suspensions of various kinds and, consequently, of various microstructures, tends to support the last statement.

2. The obvious way of obtaining expressions for mean stresses at the surface of a particle, required for formulating rheological equations of state and closing the system of equations of conservation, could consist of solving the problem of flow of an unsteady nonuniform fluid stream past a lattice of particles which at some instant are arbitrarily situated, and subsequent averaging of expressions for stresses defined by this solution over admissible configurations of particles in accordance with (1.2).

Solution of the first problem in the general case, when the concentration of particles is not small and their hydrodynamic interaction cannot be neglected, is unknown and it is doubtful if it can be solved at present. Even if particles are stationary (e.g., in a stationary granular layer) and the Reynolds number determining the flow past these is low, the no-slip conditions over a multi-connected surface of highly complex form must be satisfied. If, moreover, the particles can move, a particular "physical" nonlinearity arises, since the solution of the hydrodynamic problem substantially depends on the velocity of translational and rotational motion of all particles, while the latter are in turn determined by forces and moments acting on particles, i.e. on velocity and pressure fields in the fluid. In unsteady conditions these forces and moments also depend on the history of motion. The appearance of divergent integrals in the course of solution presents a further difficulty [8]. As shown below, these difficulties can be overcome by reversing the problem and deriving first the equations which define the flow around particles "on average". Solution of these equations in the neighborhood of a sample particle of suspension makes it possible to directly determine the unknown mean stresses.

First of all, let us formulate the equations of motion of fluid in particle interstices formally defined at all points of space occupied by the suspension. The flow of fluid, which is assumed to be incompressible, is governed by the Navier-Stokes equations

$$d_0(\partial/\partial t + \mathbf{V}\nabla)\mathbf{V} = \nabla\Sigma - d_0\nabla\Phi, \qquad \nabla\mathbf{V} = 0 \tag{2.1}$$
$$\Sigma = -P\mathbf{I} + 2\mu_0\mathbf{E}, \quad \mathbf{E} = \frac{1}{2}\left\|\frac{\partial V_i}{\partial r_j} + \frac{\partial V_j}{\partial r_i}\right\|, \quad \mathbf{I} = \|\delta_{ij}\|$$

where $\mathbf{V}(t, \mathbf{r})$ and $P(t, \mathbf{r})$ are microscopic fields of velocity and pressure (for simplicity the symbol C_N in the arguments of these functions has been omitted), and $\Phi(t, \mathbf{r})$ is the potential of external mass forces. To extend the determination of Eqs. (2.1) over the entire space we introduce function $\theta(t, \mathbf{r})$ which is equal unity at points of the fluid and vanishes inside particles. Let us consider the flow of fluid of density $d_0\theta(t, \mathbf{r})$ in which force $\theta\nabla\Sigma$ generated by stresses is acting (see also [7]). Expression (2.1) for the microscopic stress tensor $\Sigma(t, \mathbf{r})$ can be retained with the microscopic suspension velocity $\mathbf{C}(t, \mathbf{r})$, which is the same as $\mathbf{V}(t, \mathbf{r})$ outside particles and the true velocity of material inside these, substituted for $\mathbf{V}(t, \mathbf{r})$. The equations of motion of such fluid are of the form

$$d_0\theta(\partial/\partial t + \mathbf{V}\nabla)\mathbf{V} = \theta\nabla\Sigma - d_0\theta\nabla\Phi, \qquad \partial\theta/\partial t + \nabla(\theta\mathbf{V}) = 0 \tag{2.2}$$

and coincides with (2.1) in the space between particles.

Taking into consideration the properties of the averaging operation defined by (1.4) and (1.5) and that $\nabla\Phi$ is not affected by such averaging, from (2.2) we obtain the following averaged equations:

$$d_0\langle\theta(\partial/\partial t + \mathbf{V}\nabla)\mathbf{V}\rangle = \nabla\langle\theta\Sigma\rangle - \langle\Sigma\nabla\theta\rangle - d_0\langle\theta\rangle\nabla\Phi \tag{2.3}$$
$$\partial\langle\theta\rangle/\partial t + \nabla\langle\theta\mathbf{V}\rangle = 0$$

All terms of these equations must be expressed in terms of observed quantities which define the macroscopic motion of suspension. For this, using the method of [7], we obtain

$$\langle\theta\Sigma\rangle = -\langle\theta P\rangle\mathbf{I} + 2\mu_0\langle\mathbf{E}\rangle, \qquad \langle\mathbf{E}\rangle = \frac{1}{2}\left\|\frac{\partial\langle C_i\rangle}{\partial r_j} + \frac{\partial\langle C_j\rangle}{\partial r_i}\right\| \tag{2.4}$$

The definition of $\theta(t, r)$ implies that

$$\nabla\theta = \sum_{j=1}^{N}\frac{\mathbf{r}-\mathbf{r}^{(j)}}{|\mathbf{r}-\mathbf{r}^{(j)}|}\delta(|\mathbf{r}-\mathbf{r}^{(j)}| - a) = \sum_{j=1}^{N}\mathbf{n}\delta(|\mathbf{r}-\mathbf{r}^{(j)}| - a)$$

where $\mathbf{r}^{(j)}$ is the radius vector of the jth particle center, $\mathbf{n}$ is the unit vector of the external normal at the surface of particles, and summation is carried out over all particles of the system. Using this formula, the indiscernibility and statistical equivalence of particles of the ensemble, and the first of formulas (1.1), we obtain

$$\langle\Sigma\nabla\theta\rangle = \sum_{j=1}^{N}\int\varphi(t, C_N)\delta(|\mathbf{r}-\mathbf{r}^{(j)}| - a)(\Sigma\mathbf{n})\,dC_N = \tag{2.5}$$
$$N\int\varphi(t, \mathbf{r}')\delta(|\mathbf{r}-\mathbf{r}'| - a)(\langle\Sigma\rangle'\mathbf{n})\,d\mathbf{r}' =$$
$$N\int d\mathbf{a}\int d\mathbf{r}'\varphi(t, \mathbf{r}')\delta(\mathbf{r}-\mathbf{r}'-\mathbf{a})(\langle\Sigma\rangle'\mathbf{n})$$

where $\mathbf{a}$ is the vector of module a, which connects the particle center with an arbitrary point of its surface. Formula (2.5) defines the total reaction of fluid on the suspended particles. Similar terms were previously introduced in equations of the kind of (2.3) in [3 – 5] on purely physical considerations.

For the final determination of the relation between averaged terms in (2.3) and observed quantities we resort to the condition of ergodicity in the following form. We stipulate the identity of Eqs. (2.3) which define the average motion of the suspension fluid phase with similar equations derived in [1] by averaging over a small physical volume of suspension. The latter are of the form

$$d_0\varepsilon(\partial/\partial t+\mathbf{v}\nabla)\mathbf{v}=\nabla\sigma-d_0\varepsilon\nabla\Phi-\mathbf{f},\quad \partial\varepsilon/\partial t+\nabla(\varepsilon\mathbf{v})=0 \qquad (2.6)$$
$$(\varepsilon=1-\rho)$$

where $\mathbf{f}(t, \mathbf{r})$ is the mean force of interphase interaction related to a unit volume of suspension and $\sigma\ (t, \mathbf{r})$ is the mean stress tensor defined in [1] by

$$\sigma=-\varepsilon p\mathbf{I}+2\mu_0\mathbf{e}+n\int(\langle\Sigma\rangle'\mathbf{n})*\mathbf{a}da,\quad \mathbf{e}=\frac{1}{2}\left\|\frac{\partial c_i}{\partial r_j}+\frac{\partial c_j}{\partial r_i}\right\| \qquad (2.7)$$

where $\mathbf{v}(t, \mathbf{r})$, $p\ (t, \mathbf{r})$ and $\mathbf{c}(t, \mathbf{r})$ are, respectively, the averaged over the volume velocity and pressure of fluid, and of suspension [1], the asterisk denotes dyadic multiplication, and integration is carried out over the surface of a particle.

Comparison of (2.3) and (2.4) with (2.6) and (2.7) yields

$$\langle\theta\rangle=\varepsilon=1-\rho,\quad \langle\theta\mathbf{V}\rangle=\langle\theta\mathbf{C}\rangle=\varepsilon\mathbf{v},\quad \langle\mathbf{C}\rangle=\mathbf{c},\ \langle\theta P\rangle=\varepsilon p \qquad (2.8)$$
$$\langle\mathbf{E}\rangle=\mathbf{e},\quad \langle\theta(\partial/\partial t+\mathbf{V}\nabla)\mathbf{V}\rangle=\varepsilon(\partial/\partial t+\mathbf{v}\nabla)\mathbf{v}$$

Furthermore, we have the equality

$$-\langle\Sigma\nabla\theta\rangle=\nabla n\int(\langle\Sigma\rangle'\mathbf{n})*\mathbf{a}da-\mathbf{f} \qquad (2.9)$$

Note that formulas of this kind are assumed in all works on similar subjects known to the authors (see, e.g. [3 – 8]). Equalities (2.8) with the exception of the last one, are presented in an explicit form in [7]. Formula (2.9) is considered in detail below; one of its corollaries leads to the usual condition of the theory of selfconsistency. The meaning of the hypothesis about the equivalence of averaging over the ensemble and over the volume thoroughly discussed in [10].

Interesting are also the averaged equations which define (on average) the flow around a separated (sample) particle of a suspension. Such equations are derived from (2.2) by averaging over the conditional distribution function $\varphi\ (t, C_{N-1}\,|\,\mathbf{r}')$, where $\mathbf{r}'$ is the radius vector of the sample particle center. It is convenient to use in this case a system of coordinates attached to the sample particle center whose velocity in the laboratory system of measurements is $\mathbf{W}(t)$, and to assume that the Reynolds number which defines the flow is small. The linear scale of perturbations induced in the stream by the sample particle is of the order of a. Consequently in analyzing the flow around the sample particle it is possible to consider in accordance with (1.6) and (1.7) that the quantity $\rho\ (t, \mathbf{r})$ may be independent of coordinates and time, i.e. to consider the flow around a particle of a homogeneous cloud whose porosity is equal to the instantaneous value of local porosity of suspension.

The assumption of random distribution of particles, which is reflected in (1.10) leads to the following expression for $\langle\theta\rangle'$:

$$\langle\theta\rangle'=\begin{cases}\theta^*(r^*), & a<|\mathbf{r}-\mathbf{r}'|<3a\\ \varepsilon, & |\mathbf{r}-\mathbf{r}'|>3a\end{cases},\qquad r^*=|\mathbf{r}-\mathbf{r}'| \qquad (2.10)$$

Function $\theta^*\ (r^*)$ can be determined by analyzing the geometric problem of the portion of the volume of spherical layer $(r^*,\ r^*+dr^*)$, occupied by particles on the assumption of uniform distribution of particle centers in region $r > 2a$ in such a manner that this portion in that region is $\rho = 1 - \varepsilon$. Evidently, $\theta^*(a) = 1$ and $\theta^*(3a) = \varepsilon$, where ε is considered constant.

Introducing the fluid relative velocity $\mathbf{V}^*(t, \mathbf{r}) = \mathbf{V}(t, \mathbf{r}) - \mathbf{W}(t)$ and taking into account the smallness of the Reynolds number, in the coordinate system attached to the sample particle center we obtain instead of (2.2) the expression

$$d_0(\partial / \partial t)(\theta \mathbf{V}^*) = \theta \nabla \Sigma - d_0 \theta (\nabla \Phi + d\mathbf{W} / dt), \quad \partial \theta / \partial t + \nabla (\theta \mathbf{V}^*) = 0 \quad (2.11)$$

The quantity $\langle \mathbf{W} \rangle' = \mathbf{W}(t, \mathbf{r}')$ represents the mean velocity of the disperse phase at point $\mathbf{r}'$ and is independent of $\mathbf{r}$. If we introduce in addition the translational velocity $\mathbf{W}^*(t, \mathbf{r})$ of the material of particles at an arbitrary point $\mathbf{r}$ determined in the described system of coordinates, then $\langle \mathbf{W}^* \rangle' = \mathbf{w}^*(t, \mathbf{r}|\mathbf{r}') = \mathbf{w}(t, \mathbf{r}')$ defines the mean velocity of the disperse phase at point $\mathbf{r}$ in that system. It is important that the linear scale of the latter parameter which for $\mathbf{r} = \mathbf{r}'$, is zero coincides with L and, in accordance with (1.7) considerably exceeds the scale a of perturbations induced in the stream by the sample particle. Hence, when investigating the flow around the sample particle (i.e. at distances which are small in comparison with L), we can assume

$$\mathbf{U}^* = \langle \theta \mathbf{V}^* \rangle' = \langle \theta \mathbf{V}^* + (1 - \theta) \mathbf{W}^* \rangle' = \langle \mathbf{C}^* \rangle' = \langle \mathbf{C} \rangle' - \mathbf{w}(t, \mathbf{r}') \quad (2.12)$$

where $\mathbf{C}^*(t, \mathbf{r})$ is the microscopic velocity of suspension in the considered system of coordinates, and that derivatives of $\mathbf{U}^*(t, \mathbf{r})$ with respect to components of vector r are equal to corresponding derivatives of $\langle \mathbf{C} \rangle'$, i.e. that

$$\partial \mathbf{U}^* / \partial r_i = \partial \langle \mathbf{C} \rangle' / \partial r_i \quad (2.13)$$

Averaging (2.11) over the conditional distribution function $\varphi(t, C_{N-1}|\mathbf{r}')$, using (1.4) and the definition (2.12), and taking into account the relation (1.7) between scales, we obtain equations

$$d_0 \partial \mathbf{U}^* / \partial t = \nabla \langle \theta \Sigma \rangle' - \langle \Sigma \nabla \theta \rangle' - d_0 \langle \theta \rangle' \nabla \Psi, \quad \nabla \mathbf{U}^* = 0 \quad (2.14)$$
$$\Psi = \Phi + \mathbf{r} d\mathbf{w} / dt$$

where the effective potential $\Psi(t, \mathbf{r})$ of external mass forces is introduced in the considered system of coordinates. Allowing for (2.12) and (2.13) and proceeding in a similar manner, we obtain

$$\langle \theta \Sigma \rangle' = -\langle \theta \rangle' P^* \mathbf{I} + 2\mu_0 \langle \mathbf{E} \rangle', \quad \langle \theta \rangle' P^* = \langle \theta P \rangle' \quad (2.15)$$
$$\langle \mathbf{E} \rangle' = \frac{1}{2} \left\| \frac{\partial U_i^*}{\partial r_j} + \frac{\partial U_j^*}{\partial r_i} \right\| = \frac{1}{2} \left\| \frac{\partial \langle C_i \rangle'}{\partial r_j} + \frac{\partial \langle C_j \rangle'}{\partial r_i} \right\|$$
$$\langle \Sigma \nabla \theta \rangle' = (N - 1) \int \varphi(t, \mathbf{r}'' | \mathbf{r}') \delta(|\mathbf{r} - \mathbf{r}'| - a)(\langle \Sigma \rangle'' \mathbf{n}) \, d\mathbf{r}''$$

Owing to the property defined by (1.9) and relationships (2.8), (2.12) and (2.13) the asymptotic equalities

$$\lim P^* = p, \quad \lim \mathbf{U}^* = \mathbf{c} - \mathbf{w} = \varepsilon(\mathbf{v} - \mathbf{w}) = \varepsilon \mathbf{u}, \quad \lim \langle \mathbf{E} \rangle' = \mathbf{e} \quad (2.16)$$
$$\lim \operatorname{rot} \mathbf{U}^* = \operatorname{rot} \mathbf{c} \qquad (|\mathbf{r} - \mathbf{r}'| \to \infty)$$

are valid at some distance from the sample particle. In this formula $\mathbf{u}(t, \mathbf{r})$ is the fluid relative velocity averaged over the volume. These relationships define the unperturbed flow as it would have been at the point $\mathbf{r}'$ in the absence of the sample particle.

Let us consider in greater detail the expressions (2.5) for $\langle \Sigma \nabla \theta \rangle$ and (2.15) for

$\langle\Sigma\nabla\theta\rangle'$. The penultimate integral in (2.5) is a function of point $\mathbf{r}$, hence it must be calculated on condition that $\mathbf{r} = \mathrm{const}$. Actually this is an integral taken over all possible positions of centers of spherical particles such that point $\mathbf{r}$ lies on their surface [7]. To obtain a clear physical meaning of parameter $\langle\Sigma\nabla\theta\rangle$ it is expedient to expand the delta function in the integrand of the last integral in (2.5) into a Taylor series in powers of components of vector $\mathbf{a}$ and integrate termwise with respect to $\mathbf{r}'$. As the result we obtain

$$\langle\Sigma\nabla\theta\rangle = \sum_{q=0}^{\infty} \frac{(-1)^q}{q!} \int (\mathbf{a}\nabla)^q (\langle\Sigma\rangle'\mathbf{n})\, N\varphi(t, \mathbf{r})\, d\mathbf{a} \tag{2.17}$$

Allowing for (1.8), for the kth component of vector (2.17) we obtain

$$\langle\Sigma\nabla\theta\rangle_k = \sum_{q=0}^{\infty} \frac{\partial}{\partial r_l} \cdots \frac{\partial}{\partial r_m} [n(t, \mathbf{r})^q R_{l\ldots mk}(t, \mathbf{r})] \tag{2.18}$$

$${}^q R_{l\ldots mk}(t, \mathbf{r}) = \frac{(-1)^q}{q!} \int a_l \ldots a_m (\langle\Sigma\rangle'\mathbf{n})_k d\mathbf{a}$$

The series (2.18) is the expansion of total reaction, which appears in the averaged equations (2.3), in terms of multipoles, with related multipole moments defined by tensors ${}^q\mathbf{R}$, whose rank is equal to the order of multipoles. Thus the first term (a monopole) of this series defines the distributed force of interphase action in the suspension, the second (a dipole) defines the distributed couple of forces, etc. The fluid phase of suspension is thus equivalent to some homogeneous medium filling the complete space and subjected to the action of distributed force multipoles. The concept of the fictitious medium of this kind was introduced in phenomenological form in [2] and formulated more exactly in [3 – 7].

However in all these works, with the exception of [7], the problem was considered on the assumption of point forces which approximates the effect of individual particles on the flow by that of a point force equal to the interaction force between particle and fluid and applied to the latter at the point occupied by the particle center. This implies in fact the neglect of all multipole terms of series (2.18), altough it can be readily shown on the basis of computations in [5, 7] that the contribution of multipole terms in Eqs. (2.13) is comparable to that of other terms of these equations. Note that similar hypotheses about a fictitious medium were introduced in phenomenological form in investigations of effective thermal conductivity and moduli of elasticity of composite granulated materials (see review in [11]).

Vector $\langle\Sigma\nabla\theta\rangle'$ in (2.15) can be represented in the form of series of the kind of (2.18). It will be readily seen that in region $|\mathbf{r} - \mathbf{r}'| > 3a$ the constraints imposed by the presence of the sample particle at point $\mathbf{r}'$ on the possible position of other particles in accordance with (1.10) do not relate to particles which are in contact with point $\mathbf{r}$ and over which integration of (2.15) is carried out. Hence in the considered region vector $\langle\Sigma\nabla\theta\rangle'$ represents the functional of mean stresses $\langle\Sigma\rangle''$ at the surface of particles, which is of the same form as the functional of mean stresses $\langle\Sigma\rangle'$ which determines $\langle\Sigma\nabla\theta\rangle$.

However in region $a < |\mathbf{r} - \mathbf{r}'| < 3a$ the centers of spheres in contact with point $\mathbf{r}$ and over whose radius vector $\mathbf{r}''$ integration of (2.15) is carried out can only lie at points of the surface of sphere $|\mathbf{r}'' - \mathbf{r}| = a$, whose distance from the sample particle center exceeds $2a$. (This can be readily derived from formula (1.10)). It follows that the considered integration is actually extended only over the part of sphere $|\mathbf{r}''-\mathbf{r}|=a$, external to the sample particle, hence $\langle\Sigma\nabla\theta\rangle'$ is in this region explicitly dependent on $\mathbf{r}^*=\mathbf{r}-\mathbf{r}'$. Similarly to (2.10) we can write

$$\langle\Sigma\nabla\theta\rangle'=\begin{cases}\mathbf{Q}^*(\mathbf{r}^*), & a<|\mathbf{r}-\mathbf{r}'|<3a\\ \mathbf{Q}, & |\mathbf{r}-\mathbf{r}'|>3a\end{cases}\qquad(2.19)$$

where Q is a linear functional of $\langle\Sigma\rangle''$

$$\langle\Sigma\nabla\theta\rangle_k'=\sum_{q=0}^{\infty}\frac{\partial}{\partial r_l}\cdots\frac{\partial}{\partial r_m}[n(t,\mathbf{r})^q\,R'_{l\ldots mk}(t,\mathbf{r})]\qquad(2.20)$$

$${}^qR'_{l\ldots mk}(t,\mathbf{r})=\frac{(-1)^q}{q!}\int a_l\ldots a_m(\langle\Sigma\rangle''\mathbf{n})\,d\mathbf{a}$$

of a form similar to that of functional (2.18), and it is known about functional $\mathbf{Q}^*(\mathbf{r}^*)$ that for $\mathbf{r}^*=3a$ it is the same as functional (2.20) and vanishes for $\mathbf{r}^*=a$. Thus, from the phenomenological point of view, a sample particle may be considered as being submerged in the fictitious medium with distributed force multipoles and separated from the particle surface by an intermediate layer of thickness $2a$, concentric with the particle, in which the properties of the "homogeneous" medium continuously vary from those of pure fluid to those characteristic of the indicated fictitious medium. The concept of the intermediate layer of this kind along the surface of the sample particle was first introduced in [12] and considered in [7, 11].

3. Relationships derived in the preceding Section make it possible to define the linear problem of flow around the sample particle governed by Eqs. (2.14) in the form

$$d_0\partial\mathbf{U}^*/\partial t=-\nabla(\langle\theta\rangle'P^*)+\mu_0\Delta\mathbf{U}^*-\langle\Sigma\nabla\theta\rangle'-d_0\langle\theta\rangle'\nabla\Psi,\ \nabla\mathbf{U}^*=0\qquad(3.1)$$
$$\mathbf{U}^*=\boldsymbol{\lambda}\times\mathbf{r}\ \ (r=a);\ \ \mathbf{U}^*\to\mathbf{U}_0,\ \ P^*\to P_0\ \ (r\to\infty)$$

where the frame of reference origin is at the particle center, $\boldsymbol{\lambda}=\boldsymbol{\lambda}(t,\mathbf{r}'=0)$ is the mean angular velocity of rotation of the particle, $\mathbf{U}_0$ and P_0 and the derivatives of $\mathbf{U}_0$ with respect to components of $\mathbf{r}$ are expressed in terms of observed quantities in accordance with (2.16), and $\langle\theta\rangle'$ and $\langle\Sigma\nabla\theta\rangle'$ are formally defined by formulas (2.10) and (2.19), (2.20), respectively.

For simplicity of calculations we neglect in what follows the intermediate layer which separates the sample particle from the fictitious medium, whose properties are independent of $\mathbf{r}$. This approximation is of a typically operational character which, without introducing any fundamental complications, considerably simplifies computations. Physically this is equivalent to the neglect of the effect of impenetrability of spherical particles (centers of two solid spheres cannot lie closer to each other than $2a$), which is significant only at very high particle concentration. This assumption evidently limits the applicability of the developed theory to suspensions of moderate concentration. As shown by the analysis presented below and by the results cited in [7], this theory is approximately correct up to $\rho\sim 0.2-0{,}3$.

Thus we assume $\langle\theta\rangle'=\varepsilon$, $\langle\Sigma\nabla\theta\rangle'=\mathbf{Q}$. To close Eqs. (3.1) it is necessary to express in these vector $\mathbf{Q}(t,\mathbf{r})$ in terms of other unknowns. In accordance with (2.20) this vector is a linear functional of mean stresses acting at the surface of any particle of suspension in the proximity of the fixed sample particle, and the latter (stresses), owing to the linearity of equations and of problem (3.1) in general, may be considered to be linear functionals of the indicated unknowns and their derivatives at the considered, as well as at the preceding instants of time. Hence it is possible to write $\mathbf{Q}(t,\mathbf{r})=\mathbf{F}(\mathbf{U}^{**}, P^{**})$, where $\mathbf{U}^{**}(t, \mathbf{r})$ and $P^{**}(t, \mathbf{r})$ are the mean velocity and pressure, respectively, at the center of a particle in the neighborhood of the sample particle on the assumption that this particle is removed from the system, while averaging is carried out over configuration of other particles compatible with the requirement for the presence of this particle at the point. Generally these quantities are not the same as $\mathbf{U}^{*}(t, \mathbf{r})$ and $P^{*}(t, \mathbf{r})$ obtained by averaging over all possible configurations of particles on the assumption that only the position of the sample particle is fixed (see the analysis of various kinds of averaging in [7, 8]). Here we consider it possible to assume that approximately

$$\mathbf{Q}(t, \mathbf{r}) = \mathbf{F}(\mathbf{U}^{**}, P^{**}) = \mathbf{F}(\mathbf{U}^{*}, P^{*}) \tag{3.2}$$

In fact, it is possible to formulate for $\mathbf{U}^{**}(t, \mathbf{r})$ and $P^{**}(t, \mathbf{r})$ new equations which in turn depend on new conditional averages obtained by averaging over configurations in which in addition to the sample particle the position of two other particles is fixed, etc. By continuing this process, we obtain a virtually infinite ($N \gg 1$) array of equations linked by common unknowns, whose termination and closure requires some independent hypothesis. (In this connection the problem is similar to that of closure of that infinite system of moment equations in the statistical theory of turbulence). Relationship (3.2) represents the simplest version of such hypothesis, and was successfully used in this capacity previously [3 – 5, 7]. It is not difficult in principle to extend the analysis of equations of this array, using more refined hypotheses for its closure, similar to those used in the theory of turbulence or in statistical physics of fluids. The first step in this direction was made by Childress [6] who had considered equations for $\mathbf{U}^{**}$ and P^{**} in an explicit form for a steady flow in a rarefied system of particles (*). However, taking into consideration that the error associated with the use of (3.2) is not very great and has the advantage of considerable simplification of computations, the use of that formula (3.2) in the present paper is entirely justified.

To eliminate at once the effect of motion history and consider usual linear algebraic equations instead of (3.2), it is expedient to use the Fourier transform

$$\mathbf{U}^{*}(t, \mathbf{r}) = \int e^{i\omega t}\mathbf{U}^{(\omega)}(\omega, \mathbf{r})\, d\omega, \quad P^{*}(t, \mathbf{r}) = \int e^{i\omega t}P^{(\omega)}(\omega, \mathbf{r})\, d\omega, \ldots$$

Then making use of the formulated above simplifications, from (3.1) we obtain

$$\begin{aligned}
&(\mu_0\Delta - id_0\omega)\mathbf{U}^{(\omega)} - \varepsilon\nabla G^{(\omega)} - \mathbf{Q}^{(\omega)} = 0, \quad \nabla\mathbf{U}^{(\omega)} = 0, \\
&G^{(\omega)} = P^{(\omega)} + d_0\Psi^{(\omega)} \\
&\mathbf{U}^{(\omega)} = \boldsymbol{\lambda}^{(\omega)} \times \mathbf{r} \ (r = a); \quad \mathbf{U}^{(\omega)} \to \mathbf{U}_0^{(\omega)}, \quad P^{(\omega)} \to P_0^{(\omega)} \quad (r \to \infty)
\end{aligned} \tag{3.3}$$

It is possible to show on the basis of results obtained in [5, 7] and, also, those described below, that the monopole term in the expansion of $\mathbf{Q}^{(\omega)}$ derived from (2.20) has the

*) The method used in [6] was the subject of critical analysis and was further developed by Howell at the University of Cambridge and submitted for publication in the Journal of Fluid Mechanics.

form of the sum of components proportional to $\nabla G^{(\omega)}$, $\mathbf{U}^{(\omega)}$ and $\Delta\mathbf{U}^{(\omega)}$, and the dipole term has the form of the sum of terms proportional to $\nabla P^{(\omega)}$, $\Delta\mathbf{U}^{(\omega)}$ and $\Delta^2\mathbf{U}^{(\omega)}$. The quadrupole contains components proportional to $\Delta\mathbf{U}^{(\omega)}$ and $\Delta^2\mathbf{U}^{(\omega)}$, and subsequent multipole terms include components with various $\Delta^q\mathbf{U}^{(\omega)}$ $(q > 1)$ with q increasing with increasing order of the multipole. Taking into consideration that for $q > 1$ the values $\Delta^q\mathbf{U}^{(\omega)}$ can be expressed in terms of $\mathbf{U}^{(\omega)}$ and $\Delta\mathbf{U}^{(\omega)}$ taken from Eqs.(3.3) themselves, we assume below

$$\mathbf{Q}^{(\omega)} = \rho\nabla G^{(\omega)} + A\mathbf{U}^{(\omega)} + B\Delta\mathbf{U}^{(\omega)} \tag{3.4}$$

where A and B are some unknown coefficients. We introduce the apparent viscosity of the fictitious medium μ_a and coefficients of effective drag β^2 defined by

$$\mu_a = \mu_0 - B, \qquad \beta^2 = \mu_a^{-1}(id_0\omega + A) \tag{3.5}$$

and finally obtain the problem

$$(\Delta - \beta^2)\,\mathbf{U}^{(\omega)} = \mu_a^{-1}\nabla G^{(\omega)}, \quad \nabla\mathbf{U}^{(\omega)} = 0 \tag{3.6}$$

$$\mathbf{U}^{(\omega)} = \boldsymbol{\lambda}^{(\omega)} \times \mathbf{r} \quad (r = a); \quad \mathbf{U}^{(\omega)} \to \mathbf{U}_0^{(\omega)}, \quad G^{(\omega)} \to G_0^{(\omega)} \quad (r \to \infty)$$

Note that formula (3.4) can be obtained on the basis of the most general considerations. In fact, by neglecting in accordance with (3.2) the difference between $\mathbf{U}^{**}$ and P^{**}, and $\mathbf{U}^*$ and P^*, we find that the hydrodynamic situation in the neighborhood of a certain particle is defined by three vectors $\nabla G^{(\omega)}$, $\mathbf{U}^{(\omega)}$ and $\Delta\mathbf{U}^{(\omega)}$, only two of which are linearly independent (as can be readily seen from inspection of Eqs (3.3)). Because of the problem linearity we conclude that $\mathbf{Q}^{(\omega)}(\omega, r)$ can only be a linear combination of these vectors containing two indeterminate coefficients, hence formula (3.4) is valid. The coefficient at $\nabla G^{(\omega)}$ in (3.4) may be, generally speaking, chosen arbitrarily. The analysis presented below indicates that the most reasonable choice is to set that coefficient equal ρ. Coefficients A and B (or the quantities μ_a and β^2 in (3.5) and (3.6)) are so far unknown; they are subsequently determined by the ergodic condition (2.9).

The problem defined by (3.6) was solved in [3]. The importance of that solution is in that it makes possible to determine mean stresses at the surface of the sample particle by conventional methods and, thus, to calculate the integrals which were used in [1] for determining various rheological properties of suspensions.

Integrating over the surface of the sample particle, after some simple but cumbersome computations, we obtain

$$n\int \boldsymbol{\Sigma}^{(\omega)}\mathbf{n}\,da = d_0\rho\nabla\Psi^{(\omega)} + \frac{9}{2}\rho\frac{\mu_a}{a^2}\left(1 + \xi + \frac{1}{3}\xi^2\right)\mathbf{U}_0^{(\omega)} + \tag{3.7}$$
$$\frac{9}{2}\rho\frac{\mu_a}{\xi^2}\left(e^{\xi} - 1 - \xi - \frac{1}{3}\xi^2\right)\Delta\mathbf{U}_0^{(\omega)}, \quad \xi = \beta a$$

$$n\int (\boldsymbol{\Sigma}^{(\omega)}\mathbf{n}) * \mathbf{a}\,da = -\rho P_0^{(\omega)}\mathbf{I} + \frac{5\rho\mu_a}{1+\xi}\left(1 + \xi + \frac{2}{5}\xi^2 + \frac{1}{15}\xi^3\right)\mathbf{E}_0^{(\omega)} \tag{3.8}$$

where $\boldsymbol{\Sigma}^{(\omega)}$ is the Fourier transform of the stress tensor $\langle\boldsymbol{\Sigma}\rangle'$ at the surface of the sample particle and $\mathbf{E}_0^{(\omega)}$ is the Fourier transform on the strain rate tensor $\langle\mathbf{E}\rangle'$ in (2.15) at some distance from the particle. The integral in (3.7) is obviously the Fourier transform of the force acting on particles in a unit of volume in a system of coordinates attached to sample particle center.

Taking into consideration that in accordance with previous statements particle concentrations n and ρ may be assumed constant, from the ergodic condition (2.9) and formulas (2.19), (3.4), (3.7) and (3.8) we obtain

$$A = {}^9/_2\rho\mu_a a^{-2}(1 + \xi + {}^1/_3\xi^2) \tag{3.9}$$
$$B = \frac{1}{2}\rho\mu_a\left[\frac{9}{\xi^2}\left(e^\xi - 1 - \xi - \frac{1}{3}\xi^2\right) - \frac{5}{1+\xi}\left(1 + \xi + \frac{2}{5}\xi^2 + \frac{1}{15}\xi^3\right)\right]$$

Using the definition of β^2 in (3.5) and the first of formulas (3.9), for $\xi = \beta a$ we obtain the following quadratic equation:

$$\xi^2 - i\omega^\circ = {}^9/_2\rho\,(1 + \xi + {}^1/_3\xi^2), \quad \omega^\circ = \omega / \omega_*, \quad \omega_* = \mu_a\,(d_0 a^2)^{-1} \tag{3.10}$$

Equation (3.10) defines ξ as a function of ρ and of dimensionless frequency ω°. The single positive for $\omega^\circ = 0$ root of this equation is to be used. The definition of apparent viscosity μ_a in (3.5) and the second of formulas (3.9) yield the new equation

$$\alpha = \frac{\mu_a}{\mu_0} = \left[1 + \frac{9}{2}\frac{\rho}{\xi^2}\left(e^\xi - 1 - \xi - \frac{1}{3}\xi^2\right) - \frac{5\rho}{2(1+\xi)}\left(1 + \xi + \frac{2}{5}\xi^2 + \frac{1}{15}\xi^3\right)\right]^{-1} \tag{3.11}$$

The problem is thus completely determined. In the case of steady homogeneous flow the ergodic condition (2.9) becomes a condition of self-consistency which was earlier used on the basis of physical considerations in [2 – 7]. It usually yields for ξ an equation which follows from (3.10) for $\omega^\circ = 0$.

The Fourier transform of force $\mathbf{f}(t, \mathbf{r})$ and of moment $\mathbf{m}(t, \mathbf{r})$ of interphase action in a unit volume of suspension defined in the laboratory system of coordinates were actually calculated in [5]. Allowing for the difference between tensors $\langle\Sigma\rangle'$ defined in the laboratory and convective frames of reference, owing to the presence in the latter of the force of inertia, which in the laboratory system for the Fourier transform of the stress tensor $\mathbf{T}^{(\omega)}$ yields

$$\mathbf{T}^{(\omega)} = (\Sigma^{(\omega)} - d_0\Psi^{(\omega)}\mathbf{I}) + d_0\Phi^{(\omega)}\mathbf{I}$$

we obtain

$$\mathbf{f}^{(\omega)} = n\int \mathbf{T}^{(\omega)}\mathbf{n}da = d_0\rho\nabla\Phi^{(\omega)} + \frac{9}{2}\rho\frac{\mu_0}{a^2}F^{(1)}\mathbf{U}_0^{(\omega)} + \frac{3}{4}\rho\mu_0 F^{(2)}\Delta\mathbf{U}_0^{(\omega)} \tag{3.12}$$

$$\mathbf{m}^{(\omega)} = n\int (\mathbf{T}^{(\omega)}\mathbf{n}) \times \mathbf{a}da = 6\rho\mu_0\,(M^{(1)}\,{}^1/_2\operatorname{rot}\mathbf{U}_0^{(\omega)} - M^{(2)}\boldsymbol{\lambda}^{(\omega)}) \tag{3.13}$$

where the coefficients

$$F^{(1)} = \alpha\,(1 + \xi + {}^1/_3\xi^2), \quad F^{(2)} = 6\alpha\xi^{-2}\,(e^\xi - 1 - \xi - {}^1/_3\xi^2) \tag{3.14}$$
$$M^{(1)} = \alpha\,(1 + \xi)^{-1}e^\xi, \quad M^{(2)} = \alpha\,(1 + \xi)^{-1}\,(1 + \xi + {}^1/_3\,\xi^2)$$

become equal to unity when $\xi \to 0$ and, consequently, expressions (3.12) and (3.13) for $\rho \to 0$ become the known formulas for the force and moment acting on n noninteracting particles in a harmonic pulsating flow.

For the Fourier transform of tensor $\sigma(t, \mathbf{r})$ from (2.7) and (3.8) we have

$$\sigma^{(\omega)} = -P_0^{(\omega)}\mathbf{I} + 2\mu\mathbf{E}_0^{(\omega)} \tag{3.15}$$

where we have introduced the coefficient of effective viscosity

$$\mu = \mu_0 \, [1 + {}^5/_2 \rho\alpha \, (1 + \xi)^{-1} \, (1 + \xi + {}^2/_5 \xi^2 + {}^1/_{15} \xi^3)] \tag{3.16}$$

Finally, for the Fourier transform $\chi^{(\omega)}$ of the moment stress tensor $\chi \, (t, \, \mathbf{r})$, averaged over the volume and determined in [1], after computations we obtain

$$\chi^{(\omega)} + \gamma^{(\omega)} = n \int (\varepsilon \mathbf{aa}) * (\mathbf{T}^{(\omega)} \mathbf{n}) \widetilde{d\mathbf{a}} = 2\eta \mathbf{Y}_0^{(\omega)}, \quad \varepsilon = \| \varepsilon_{ijk} \|, \quad \gamma = \frac{a^2}{5} \| \varepsilon_{ikj} f_j \| \tag{3.17}$$

where ε is the alternating Levi-Civita tensor, and tensor

$$\mathbf{Y}_0^{(\omega)} = \frac{1}{2} \left\| \frac{\partial \, (\mathrm{rot} \, \mathbf{U}_0^{(\omega)})_i}{\partial r_j} + \frac{\partial \, (\mathrm{rot} \, \mathbf{U}_0^{(\omega)})_j}{\partial r_i} \right\|$$

is the Fourier transform of tensor

$$\mathbf{y} = \frac{1}{2} \left\| \frac{\partial \, (\mathrm{rot} \, \mathbf{c})_i}{\partial r_i} + \frac{\partial \, (\mathrm{rot} \, \mathbf{c})_j}{\partial r_i} \right\| \tag{3.18}$$

where relationsips (2.16) are explicitly taken into account. In (3.17) we have also introduced the effective coefficient of "moment" viscosity

$$\eta = a^2 \mu_0 \alpha \rho \, (1 + \xi + {}^1/_3 \xi^2)^{-1} e^{\xi} \tag{3.19}$$

Note that calculations actually result in the appearance in (3.15) and (3.17) of additional terms proportional to $\Delta \mathbf{E}_0^{(\omega)}$ and $\Delta \mathbf{Y}_0^{(\omega)}$, respectively. Taking into account that the scale of $\mathbf{E}_0^{(\omega)}$ and $\mathbf{Y}_0^{(\omega)}$ is equal L, it is not difficult to show that the ratio of such terms to corresponding components in (3.15) and (3.17) with $\mathbf{E}_0^{(\omega)}$ or $\mathbf{Y}_0^{(\omega)}$ is of the order of $(a \, / \, L)^2$, i.e. such terms can be neglected.

Parameters α and ξ appearing in the above formulas are defined by (3.15) and (3.17). It will be readily seen that these and the introduced rheological coefficients (3.14), (3.16) and (3.19) depend on the stream frequency ω, i.e. that frequency dispersion of these coefficients takes place. Thus $\mathbf{f} \, (t, \, \mathbf{r})$, $\mathbf{m} \, (t, \, \mathbf{r})$, $\sigma \, (t, \, \mathbf{r})$ and $\chi \, (t, \, \mathbf{r})$ which are defined as the inverse Fourier transforms of (3.12), (3.13), (3.15) and (3.17), respectively, represent in the general case some functionals of the velocity field unperturbed by the sample particle, and of its derivatives.

The frequency dispersion is, however, important only for $\omega^\circ \gtrsim 1$ $(\omega \gtrsim \omega_*)$. The characteristic frequency ω_* in (3.10) is usually very high, hence in practically important cases $\omega^\circ \ll 1$, and the indicated frequency dispersion can be generally neglected, introducing "quasi-stationary" rheological coefficients obtained from (3.14), (3.16) and (3.19) for $\omega^\circ = 0$.

Note also that for ${}^1/_2 \mathrm{rot} \mathbf{U}_0^{(\omega)}$ and $-\lambda^{(\omega)}$ the coefficients defined by (3.14) and appearing in (3.13) are different, although they become asymptotically the same when $\xi \to 0$, i.e. in the case of diluted suspensions. This somewhat unexpested but theoretically important result which shows that phenomenological attempts at equating these coefficients made in many known works are not sufficiently substantiated.

4. Let us consider in more detail a flow whose characteristic frequency satisfies the condition $\omega^\circ \ll 1$. From Eq. (3.10) we then have

$$\xi = \frac{3}{2 \, (2 - 3\rho)} \, [3\rho + (8\rho - 3\rho^2)^{1/2}] \tag{4.1}$$

which coincides with the result obtained by Brinkman [2], Tam [3] and others. The apparent viscosity μ_a (or coefficient α) is determined by (3.11). Since the latter is a function of ρ it increases with ρ up to $\rho \approx 0.2$ and for $\rho > 0.3$ it rapidly falls with increasing ρ. This function which was determined above by using the ergodic condition (2.9) is the same as the similar function determined in [7] by direct computation of the integral appearing in its definition in (2.15). The decrease of μ_a with increasing ρ in the region $\rho > 0.2$ is explained by the rapid increase in that region of the component of interphase action force (3.7) which is proportional to the Laplacian of velocity which increases coefficient B in (3.9).

Allowing for (2.16) and conditions for $r \to \infty$ in (3.1) and (3.3), and using the last formulas in Sect. 3, we obtain the following rheological equations of state:

$$\mathbf{f}(t, \mathbf{r}) = d_0\rho\nabla\Phi + \frac{9}{2}\rho\frac{\mu_0}{a^2}F^{(1)}\varepsilon\mathbf{u} + \frac{3}{4}\rho\mu_0F^{(2)}\Delta\mathbf{c} \qquad (4.2)$$

$$\mathbf{m}(t, \mathbf{r}) = 6\rho\mu_0(M^{(1)}\,{}^1\!/_2\,\mathrm{rot}\,\mathbf{c} - M^{(2)}\boldsymbol{\lambda})$$

$$\boldsymbol{\sigma}(t, \mathbf{r}) = -p\mathbf{I} + 2\mu\mathbf{e}, \qquad \boldsymbol{\chi}(t, \mathbf{r}) = 2\eta\mathbf{y} - \boldsymbol{\gamma}$$

Tensors $\mathbf{e}(t, \mathbf{r})$ and $\mathbf{y}(t, \mathbf{r})$ in (4.2) are defined by (2.7) and (3.18), respectively, and the coefficients $F^{(i)}$ and $M^{(i)}$ $(i = 1,2)$ and of viscosity μ and η are represented in (3.14), (3.16) and (3.19) as functions of ξ and α defined, respectively, by (4.1) and (3.11).

Let us write down the equations of motion of suspension as formulated in [1] but with allowance for relationships (4.2) which are valid for $\omega^\circ \ll 1$. We have

equations of conservation of mass of phases

$$\partial\varepsilon/\partial t + \nabla(\varepsilon\mathbf{v}) = 0, \qquad \partial\rho/\partial t + \nabla(\rho\mathbf{w}) = 0 \qquad (4.3)$$

equations of conservation of momentum of phases

$$d_0\varepsilon(\partial/\partial t + \mathbf{v}\nabla)\mathbf{v} = -\nabla p + 2\nabla(\mu\mathbf{e}) - {}^3\!/_4\,\rho\mu_0F^{(2)}\Delta\mathbf{c} - {}^9\!/_2\,\rho\mu_0a^{-2}F^{(1)}\varepsilon(\mathbf{v} - \mathbf{w}) - d_0\nabla\Phi \qquad (4.4)$$

$$d_1\rho(\partial/\partial t + \mathbf{w}\nabla)\mathbf{w} = {}^3\!/_4\,\rho\mu_0F^{(2)}\Delta\mathbf{c} + {}^9\!/_2\,\rho\mu_0a^{-2}F^{(1)}\varepsilon(\mathbf{v} - \mathbf{w}) - (d_1 - d_0)\rho\nabla\Phi$$

equations of conservation of moment of momentum of phases

$$d_0\varepsilon(\partial/\partial t + \mathbf{v}\nabla)\mathbf{K}_0 = 2\nabla(\eta\mathbf{y}) - 6\rho\mu_0(M^{(1)}\,{}^1\!/_2\,\mathrm{rot}\,\mathbf{c} - M^{(2)}\boldsymbol{\lambda}) - \nabla\boldsymbol{\gamma} \qquad (4.5)$$

$${}^2\!/_5\,a^2d_1\rho(\partial/\partial t + \mathbf{w}\nabla)\boldsymbol{\lambda} = 6\rho\mu_0(M^{(1)}\,{}^1\!/_2\,\mathrm{rot}\,\mathbf{c} - M^{(2)}\boldsymbol{\lambda})$$

where $d_0\mathbf{K}_0(t, \mathbf{r})$ is the mean internal moment of momentum of the continuum simulating the fluid phase of suspension, and the similar parameter for the disperse phase is ${}^2\!/_5a^2d_1\boldsymbol{\lambda}(t, \mathbf{r})$ [1].

Equations defining particular kinds of flow are readily obtained from (4.3) – (4.5). For example, filtration of fluid through a stationary granulated layer is governed by the first of Eqs. (4.3) – (4.5) with $\mathbf{c} = \varepsilon\mathbf{v}$, $\mathbf{w} = 0$ and $\boldsymbol{\lambda} = 0$. In this case the effective viscosity of the filtrated fluid is $\mu - {}^3\!/_4\rho\mu_0F^{(2)} = \mu_a$, i.e. it is equal to the apparent viscosity of the fictitious medium considered above. Equations for the one-velocity model of a "homogeneous" suspension are similarly analyzed. The equation of conservation of momentum for such suspension is obtained by term-by-term addition of Eqs. (4.4), which shows that the effective viscosity is in this case equal μ.

Note that the rheological properties of the considered disperse system is wholly determined by stresses which appear on the surface of the system particles, which in turn depend only on the nature of the relative flow of fluid and not on particular features of the absolute phase motion. It follows from the presented analysis that the rotation of particles has no effect whatsoever on the formation of rheological properties of continua which simulate suspension phases. Hence, in particular, there is no basic difference between the problem of fluid filtration through a granulated layer of stationary particles and that of flow of the suspension fluid phase in the sense that the rheological properties of both kinds of flows equally depend on the nature of the fluid relative motion, and when the latter are the same, these properties are, also, the same. This conclusion, which here appears to be an obvious corollary of the developed theory, contradicts the widely held view on the essential qualitative difference between the two problems.

The apparent confirmation of the latter point of view by the work of Lundgren [7] is the result of the unnatural choice of initial representation for parameter Q substantially different from (3.4) for investigating the flow of suspension in a gravity field and the subsequent use of the averaged equation of conservation of momentum of the fictitious medium flowing past the sample particle, which differs from the equation analogous to (3.1) by a certain multiplier. This resulted in the incorrect determination in [7] of the mean stress tensor in the fluid flowing past the sample particle so that, for instance, the apparent viscosity of that fluid is not the same as the analogous quantity in the equation of conservation of momentum of the fluid phase, which is derived by averaging over a small physical volume. The incorrect conclusion about the difference between the interphase interaction force $\mathbf{f}(t, \mathbf{r})$ in a suspension and a stationary granulated layer, derived on this basis, was used in [7] for explaining the effect of reduced hydraulic resistance of a pseudo-fluidized layer in comparison with that of a stationary granulated layer of the same porosity. Such explanation is evidently false ; the "regular" force $\mathbf{f}(t, \mathbf{r})$ is, in fact, in both cases the same, and the effect of lower resistance is apparently explained by appearance in the quasi-fluidized layer of an additional nonzero interaction force induced by local porosity fluctuations in accordance with the model described in [13, 14].

For $\rho \ll 1$ (3.16) with allowance for (3.11) and (4.1) yields the known result obtained by Einstein

$$\mu = \mu_0 (1 + {}^5/_2\rho) \tag{4.6}$$

We note in this connection that Pokrovskii [15] proved the inaccuracy of this result and suggested the alternative formula

$$\mu = \mu_0 (1 + {}^3/_2\rho) \tag{4.7}$$

which contradicts experimental data, and was subjected to deserved criticism (see, e.g. remarks in review [16]). Since this conclusion was repeated in subsequent works by Pokrovskii, we would point out the essence of the error made by him. He considered the conventional method of derivation of formula (4.6) incorrect because in it the tensor

$${}^1/_2 \, \| \partial v_i / \partial r_j + \partial v_j / \partial r_i \| \tag{4.8}$$

was used as the averaged over the (observed) volume tensor of the suspension rate of strain $\mathbf{e}(t, \mathbf{r})$. He proposed, without any substantiation, to use that tensor multiplied by the suspension porosity $\varepsilon(t, \mathbf{r})$. This contradicts the strict result given by (2.7) which is always correct, except for $\mathbf{w} = 0$ (i.e. the case of fluid filtration through a stationary layer). It is at the same time clear that to consider effective viscosity as the parameter defining the suspension as a whole has any meaning only in the opposite limit case of $\mathbf{w}(t, \mathbf{r}) \approx \mathbf{v}(t, \mathbf{r})$, when it is possible to consider a suspension as an approximation to a certain homogeneous medium. In the latter case tensor $\mathbf{e}(t, \mathbf{r})$ in (2.7) is the same as that defined by (4.8), as is usally tacitly assumed in the derivation of (4.6). Furthermore, for $\mathbf{w} = 0$, when the considerations of the rate of strain tensor in [15] are valid, formula (4.7) has no meaning whatsoever, since in that case the motion is defined by viscosity μ_a and not μ.

In concluding we shall briefly discuss the assumptions made above. The first group of assumptions (on the relationships (1.6) and (1.7) between characteristic scales, on the insignificant effect of individual particles on the flow at distant points in accordance with (1.9), and on the smallness of the Reynolds number of the flow around particles) defines a fairly wide class of systems and their motions to which the proposed theory is applicable.

The second group comprises three assumptions aimed at the simplification of reasoning and computation. The first of these relates to the random distribution of particles in the neighborhood of any separated particle, as reflected in (1.10). Such assumption is evidently valid for chaotically packed stationary granulated layers but is not usually satisfied in the case of suspensions. However it follows from the analysis in [16] that the effect of the exact form of the binary distribution function which defines preferred configurations of particle pairs in the stream is in many cases insignificant, which justifies the use of that assumption. To reject it, it is necessary, in the first instance, to consider the interaction between two suspended particles in streams of various kinds, as was done in [9] for particles in a simple shear stream and then use a new relationship of the kind of (1.10) into which is introduced the binary function of distribution obtained by such analysis.

Closely related to this is the second assumption about neglecting the effect of particle impenetrability with consequent assumption of the existence of a transition layer around the surface of the sample particle, in which the properties of the fictitious medium substantially depend on coordinates. It follows from the above consideration and the comparison of the obtained results with experimantal data that this assumption is approximately valid for moderately concentrated suspensions with ρ not exceeding 0.2 – 0.3. The rejection of this assumption is related to a considerably more complex mathematical problem of flow around the sample particle; it does not, however, introduce any fundamental difficulties.

Finally, the assumption was made that the mean velocity and pressure at some point of fluid is approximately the same as the conditional means at that point, obtained by averaging only over admissible particle configurations as they would be if the center of one of the particles were located at the considered point. As previously stated this is the simplest hypothesis which makes it possible to close the theory. Although the analysis of the limits of validity of this hypothesis and of the magnitude of the error introduced by it is very cumbersome, it can be carried out by the method described in [6].

We note that, owing to the assumption of smallness of the Reynolds number, the effect of particle pulsation in the fluid on the formation of rheological properties of suspension is neglected here and in [1], although, theoretically, in a number of cases that effect may be substantial. We also note that all results obtained here define the suspension at some distance from its boundaries, hence the problem of adequate boundary conditions which must be imposed at surfaces of various kinds remains open.

Let us briefly point out possible ways for extending developed here theory. First of all, it is not difficult to extend it to emulsions of drops or bubbles whose shape is close to spherical. The first step in this direction was taken in [5] by considering the interaction between phases of an emulsion. The extension of this theory to suspensions of particles with asymmetric properties is also of great interest. Such are, in the first instance, suspensions of spheres with dipole moments in the related external field (e.g. suspension of magnetized particles in the presence of a magnetic field or particles with offset center of gravity in a gravitational field), as well as suspensions of spheroidal or ellipsoidal particles.

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A METHOD OF EXAMINING A PAIR OF INTERDEPENDENT INTEGRAL EQUATIONS

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The paper deals with the method of inverting two singular integral equations of the first and second kind, respectively, possessing a definite structure. The equations as well as their solutions are obtained on the basis of analyzing a specific mixed problem of the potential theory for a quadrant.

1. Let us seek two functions, $\varphi_1(z)$ and $\chi_1(z)$ regular in the upper right quadrant, vanishig at infinity and satisfying the following conditions at the boundary rays:

$$\varkappa\varphi_1(t)+\overline{\chi_1(t)}=f_1(t),\quad t=x,\quad 0\leqslant x<\infty \tag{1.1}$$

$$\varphi_1(t)+\overline{\chi_1(t)}=f_2(t);\quad t=iy,\quad 0\leqslant y<\infty \tag{1.2}$$

Generally speaking, $\varkappa$ is a complex parameter and the specified functions $f_1(t)$ and $f_2(t)$ satisfy the Hölder's condition and are of the order $O(1/t)$ at infinity. In what follows we shall assume, without loss of generality, that $f_2(t)=0$. We arrive at this case by subtracting from the solution which is being sought, the particular solution for the right semi-plane with the condition (1.2) holding along its whole boundary (and in particular, when $f_2(t)$ is zero on the negative half of the ordinates).

Let us introduce the auxilliary function $\omega(t)$ on the ray $(0\leqslant x<\infty)$

$$A\varphi_1(t)-\overline{\chi_1(t)}=2\omega(t),\qquad 0\leqslant t<\infty \tag{1.3}$$

where A is a certain complex constant. Adding and subtracting (1.1) and (1.3) term by term, we obtain

$$\varphi_1(t)=\frac{1}{A+\varkappa}[2\omega(t)+f_1(t)] \tag{1.4}$$

$$\chi_1(t)=\frac{1}{\bar A+\bar\varkappa}[-2\bar\varkappa\,\overline{\omega(t)}+\bar A\overline{f_1(t)}]$$

Let us now define new functions $\varphi(z)$ and $\chi(z)$ regular in the upper right quadrant

$$\varphi(z)=\varphi_1(z)-\frac{1}{A+\varkappa}\frac{1}{\pi i}\int_0^\infty\frac{\omega(t)}{t-z}dt-\frac{1}{A+\varkappa}\frac{1}{2\pi i}\int_0^\infty\frac{f_1(t)}{t-z}dt \tag{1.5}$$

$$\chi(z)=\chi_1(z)+\frac{\bar\varkappa}{\bar A+\bar\varkappa}\frac{1}{\pi i}\int_0^\infty\frac{\overline{\omega(t)}}{t-z}dt-\frac{\bar A}{\bar A+\bar\varkappa}\frac{1}{2\pi i}\int_0^\infty\frac{\overline{f_1(t)}}{t-z}dt$$